R. C. Huffer

On Methods for Determining Complex Roots of Algebraic Equations

ON METHODS FOR DETERMINING COMPLEX ROOTS OF ALGEBRAIC EQUATIONS

BY

RALPH CRAIG HUFFER
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I HEREBY RECOMMEND THAT THE THESIS PREPARED UNDER MY SUPERVISION BY Ralph Craig Huffer

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Aubrey J. Kempner
In Charge of Thesis

E. J. Townsend
Head of Department

Recommendation concurred in*

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Committee on Final Examination*

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CHAPTER I.

INTRODUCTION.

CHAPTER I.

INTRODUCTION.

1.Object of the Thesis.

There are a large number of methods for determining to any desired degree of accuracy the complex roots of numerical equations. The problem has been attacked by many of the foremost mathematicians and many methods of approximation have been developed.This thesis aims to present and to discuss the more important of these methods, stating the relations between the various ones wherever possible.

This work will deal only with equations whose coefficients are real.Though some of the methods treated can be applied directly to equations with complex coefficients,it is not the purpose of this article to discuss this application in detail.The problem for equations with complex coefficients is not more general,but leads to equations of higher degree with real coefficients,it being merely necessary to multiply the original function by the conjugate function in order to make all coefficients real.

No attempt will be made to go carefully into the graphical methods for determining complex roots.The algebraic methods will be adhered to except in those cases in which a graphical representation will make clear the algebraic processes in use.

An outline of the study of complex roots has been given by F.Cajori in an article entitled "A History of the Arithmetical Methods of Approximation to the Roots of Numerical Equations" published in the Colorado College Publication of November,1910.

The article deals primarily with the history of the study of all roots, but those parts of it regarding complex roots have been of value in the preparation of this thesis. The references contained in it have been especially valuable.

A list of papers dealing with this subject or with phases of it is given at the end of this thesis.

2. Historical Outline.[5,6,7,8,9]

The history of the development of methods for the determination of complex roots of algebraic equations is so closely bound up with that of real roots that the separation of the two is rather difficult. Many of the methods which apply to complex roots were discovered in the search for a means of determining real roots, though some of them, devised originally for real roots, were not applied to complex roots until many years after their discovery, as, for example, the methods of Newton[56] and of Horner.[63]

The researches of Budan[3,6] and Fourier[26,27,6] in the first quarter of the nineteenth century, and later the theorems of Sturm[69,6] in 1835, of Cauchy[11,12,13] and of Laguerre[39,6] laid a foundation for work on complex roots. The method of Newton, first developed in 1671, was applied to complex roots by Simpson[66,67] about 1750 and by Cauchy[11] in 1849.* The method of Taylor's theorem, proposed[7] for real roots in 1717, was not applied to complex roots before the ninteenth century.[6]

In 1728 Daniel Bernoulli[2,6] # suggested the use of recurrent series for finding roots of equations and this formed a basis for

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*Waring is credited by Cajori,[6 p. 237] with having first extended Newton's method to complex roots, but no reference is given.
#This reference was not available. See Cajori,[6,7]

work done later by Euler[24] and Lagrange[38] in developing what is now called Bernoulli's method,and,of still greater importance,for the method worked out by Gräffe[31] in 1837.

Since the first part of the ninteenth century there has been a great interest in the determination of complex roots which has resulted in the developement of a large number of new methods as well as the extension to complex roots of methods for finding real roots.In 1870 Schröder[64] made an attempt to unite a number of these into one general method.His results,while perhaps of little practical value,are useful theoretically in that they show the relationship between several methods;while the article is at least a step towards the generalization of methods of attacking the problem.

3.General Classification.

The methods of approximation may be divided into three classes,as follows:

1.Methods of repeated limits(with rectangular coordinates).

2.Methods of repeated limits(with polar coordinates).

3.Methods of double limits.

In the first class the root will be expressed in the form $\alpha+i\beta$.It will be found by means of two distinct limiting processes; the one will determine the real part of the root α,the other the imaginary part $i\beta$. No matter whether α or β is to be first determined,it will be definitely fixed before any step is taken to find the other,except in some cases for a rough preliminary location of the complex roots which precedes the more accurate

determination. This method of repeated limits may be expressed as:

$$\underset{\beta' \doteq \beta}{L}\ \underset{\alpha' \doteq \alpha}{L}\ (\alpha' + i\beta') = \alpha + i\beta,$$

or

$$\underset{\alpha' \doteq \alpha}{L}\ \underset{\beta' \doteq \beta}{L}\ (\alpha' + i\beta') = \alpha + i\beta.$$

where $\alpha' + i\beta'$ is an approximation to the root $\alpha + i\beta$

The second class is similar to the first, except that polar coordinates are used; thus:

$$\underset{\theta' \doteq \theta}{L}\ \underset{\rho' \doteq \rho}{L}\ \rho' e^{i\theta'} = \rho e^{i\theta}$$

or

$$\underset{\rho' \doteq \rho}{L}\ \underset{\theta' \doteq \theta}{L}\ \rho' e^{i\theta'} = \rho e^{i\theta}.$$

where $\rho' e^{i\theta'}$ is an approximation to the root $\rho e^{i\theta}$.

The third class contains those methods making use of double limits; that is, neither α nor β (nor ρ nor θ) is definitely determined before the other, but successive approximations give improved values for both α and β (or ρ and θ); thus:

$$\underset{\substack{\alpha' \doteq \alpha \\ \beta' \doteq \beta}}{L}\ (\alpha' + i\beta') = \alpha + i\beta$$

or

$$\underset{\substack{\rho \doteq \rho' \\ \theta \doteq \theta'}}{L}\ \rho' e^{i\theta'} = \rho e^{i\theta}.$$

This method of classification is very convenient for the purposes of this paper, as it includes all cases that will be considered. There are cases in which the complex number is not broken into two separate parts, such as certain graphical methods or methods of vectors. These would of course not come under this classification. It is the object of this thesis, however, to deal only with algebraic methods, and all of these fall very conveniently into the classes as stated above.

CHAPTER II.

METHODS OF REPEATED LIMITS. (RECTANGULAR COORDINATES).

4. Lagrange's Method. 15, 19, 21, 31, 68, 44, 45, 38.

Given an equation

$$f(z) = 0,$$

of the nth degree, with real coefficients, having two or more complex roots $(\alpha + i\beta), (\alpha - i\beta)$. Form the equation whose roots are equal to the squares of the differences of pairs of roots of the given equation. The difference of two conjugate roots is $2i\beta$ which, when squared, gives $-4\beta^2$. The square of the difference of two real roots will be a real, positive number. The square of the difference of a real and a complex number, or of two non-conjugate complex numbers will in general give a complex number. Hence to each real, negative root w of the new equation will correspond two conjugate, complex roots of the original equation. Also, since $w = -4\beta^2$, the imaginary parts of these two roots will be equal to $\frac{-i\sqrt{-w}}{2}, \frac{i\sqrt{-w}}{2}$.

Having determined β, substitute $\alpha + i\beta$ for z in $f(z) = 0$, obtaining two equations in

$$u(\alpha) = 0, \ v(\alpha) = 0,$$

where u includes all the terms of $f(\alpha + i\beta)$ that are real, while v includes only those involving i. The correct value of α that corresponds to β is obtained by finding the common root of $u = 0$ and $v = 0$. This can be done by means of rational operations, it being equivalent to finding the greatest common divisor of

the functions u and v.

There are two cases which require special attention. First, if a real root is equal to the real part of a complex root, the difference between the two will be the imaginary part of the complex root; and the square of this difference will be a negative number. Second, if two complex roots have equal real parts but the roots are non-conjugate, the squared difference will again be a negative number. In either case we will have a negative root of the equation of squared differences to which two conjugate complex roots do not correspond.[56]

To examine the first case, say that the complex roots are $(\alpha+i\beta), (\alpha-i\beta)$, and the real root α. Then the equation of squared differences will have ~~two~~ roots equal to $-\beta^2$; but to ~~two of~~ these will not correspond two conjugate complex roots each.[77]

In the second case, let the roots be $(\alpha+i\beta), (\alpha-i\beta); (\alpha+i\gamma), (\alpha-i\gamma)$. Then the equation of squared differences will have as roots $-4\beta^2, -4\gamma^2$,

$$(i\beta+i\gamma)^2=(-\beta^2-2\beta\gamma-\gamma^2) \quad , \quad (i\beta-i\gamma)^2=(-\beta^2+2\beta\gamma-\gamma^2)$$

$$(-i\beta+i\gamma)^2=(-\beta^2+2\beta\gamma-\gamma^2) \quad , \quad (-i\beta-i\gamma)^2=(-\beta^2-2\beta\gamma-\gamma^2).$$

Here are two pairs of equal roots to which do not correspond two conjugate roots. To the single roots $-4\beta^2, -4\gamma^2$, however, there are corresponding pairs of conjugate roots.

Lagrange[38,77] deduces the following conclusions: (1) When all the real negative roots of the equation of the squares of the differences of the roots of the original equation are unequal, the original equation will have a pair of conjugate complex roots corresponding to each one. (2). If among the negative roots

of the squared differences equation equal roots are found, then each unequal root will always furnish a pair of imaginary roots. Each pair of equal roots, however, may give two pairs, or no, imaginary roots. Three equal roots, from similar reasoning, may give six, or two; four may give eight, four, or none; and so on. In these latter cases, the exact number of complex roots corresponding to equal negative roots of the equation of squared differences can be determined when $u = 0$ and $v = 0$ are solved for α. For example, if we get two equal roots w, $u = 0$ and $v = 0$ must give two equal values of α. If these two values are real, the equation has two pairs of complex, conjugate roots corresponding to the two equal values of w; if complex, none.

In theory, Lagrange has developed a very clear and exact method for the determination of the complex roots of an equation; from the practical side, however, it is almost without value, at least for equations of higher degree than the fourth. The process of building up the equation of squared differences, while possible by means of the symmetric functions of the roots, becomes very complicated; and even if one could obtain it easily, it would be of degree $\frac{n(n-1)}{2}$ which for n greater than 4 becomes so large as to render the determination of it's real roots very tedious.

*Example: Find the complex roots of

$$x^4 + x^3 - 5x^2 - 7x + 10 = 0.$$

The equation of the squared differences is

$$x^6 - 43x^5 - 679x^4 - 4209x^3 - 248x^2 + 99724x - 96400 = 0.$$

Solving this equation for real roots, we find them to be -4, and 1. We have therefore one pair of complex roots of the original

*The examples given are solved by several methods where practical.

equation whose imaginary part

$$\beta = \pm \frac{i\sqrt{-w}}{2} = \pm \frac{i\sqrt{4}}{2} = \pm i.$$

Substituting $(\alpha + i\beta)$ for x in the original equation, and setting $\beta = 1$, we have

$$\alpha^4 + \alpha^3 - 11\alpha^2 - 10\alpha + 16 = 0$$
$$4\alpha^3 + 3\alpha^2 - 14\alpha - 8 = 0$$

The greatest common divisor of the left-hand sides is $\alpha + 2$. Hence -2 is the root sought, and the two desired roots of the original equation are $-2+i$, $-2-i$. Had we placed $\beta = -1$ in the above equations, the resulting roots α would have been the same, which would be due to the fact that β enters to even powers only in the two equations.

5. Method based upon Taylor's Theorem. [1, 20, 21, 58, 77, 31, 65, 63.]

Given an equation of the nth degree with real coefficients

$$f(z) \equiv f(x+iy) = 0,$$

where x and y are real. Expand by means of Taylor's theorem, thus:

$$f(x+iy) \equiv f(x) + \frac{f'(x)\cdot iy}{1!} + \frac{f''(x)\cdot i^2y^2}{2!} + \cdots + \frac{f^{(n)}(x)\cdot i^ny^n}{n!}.$$

Since the condition that $f(x+iy) = 0$ is that the sum of the real terms and the sum of the imaginary terms must each be equal to zero, we have

$$f(x) - \frac{f''(x)\cdot y^2}{2!} + \frac{f''''(x)\cdot y^4}{4!} - \ldots = 0,$$

$$\frac{f'(x)}{1!} - \frac{f'''(x)\cdot y^2}{3!} + \frac{f^{v}(x)\cdot y^4}{5!} - \ldots = 0,$$

where each equation has a finite number of terms, and is of degree not greater than n in y; or, since only even powers of y

appear, of degree not greater than $\frac{n}{2}$ in y^2.

Eliminating y from these two equations, we can solve the resulting equation for real values of x, which we can substitute in one of the above equations to obtain y. The equation in x will have $\frac{n(n-1)}{2}$ roots, each equal to one-half the sum of a pair of roots of $f(z) = 0$*Had we eliminated x, the resulting equation in y^2 would have had roots equal to $-\frac{1}{4}$ the square of the differences of pairs of roots of $f(z) = 0$. To determine roots of $f(z) = 0$, we need consider only real values of x and y, all such pairs for which $y \neq 0$ giving complex roots.

This method is evidently closely associated with that of Lagrange, and is subject to much the same limitation. The process of elimination between the ~~the~~ two equations is very complicated, leading as in the former case to a solution of an equation of degree $\frac{n(n-1)}{2}$; in fact, if x is the variable that is eliminated, the equations to be solved have identical roots except for a constant factor. Hence this method has little, if any, advantage over that of Lagrange if judged from the practical standpoint. Where one method demands the formation of the equation of squared differences and the determination of the greatest common divisor of two functions, the other requires the elimination of a variable between two equations and later the solution of an additional equation for it's real roots.

The two equations in x and y could have been obtained by a substitution of $x + iy$ for z, and the separation made as before. The

* See Bauer,' p.239.

problem becomes the same throughout. 1, 31, 65, 63.

Example: $z^4 + z^3 - 5z^2 - 7z + 10 = 0.$

By Taylor's Theorem,

$$f(x+iy) = f(x) + \frac{f'(x)\cdot iy}{1!} - \frac{f''(x)\cdot y^2}{2!} - \frac{f'''(x)\cdot iy^3}{3!} + \frac{f''''(x)\cdot y^4}{4!} = 0.$$

or, $$f(x) - \frac{f''(x)\cdot y^2}{2!} + \frac{f''''(x)\cdot y^4}{4!} = 0.$$

$$\frac{f'(x)}{1!} - \frac{f'''(x)y^2}{3!} = 0.$$

These give us the two equations

$$(x^4 + x^3 - 5x^2 - 7x + 10) - (6x^2 + 3x - 5)y^2 + y^4 = 0.$$

$$(4x^3 + 3x^2 - 10x - 7) - (4x + 1)y^2 = 0.$$

Eliminating y from these two equations, we have after simplifying

$$16x^6 + 24x^5 - 28x^4 - 38x^3 - 32x^2 - 11x - 6 = 0.$$

The roots of this equation are -2, $\frac{3}{2}$, and four imaginary roots.

Substituting these values of x in the second equation in x and y, we find for $x = -2$, $y = \pm 1$. Therefore the complex roots of the equation are $-2 + i$, and $-2 - i$. For $x = \frac{3}{2}$, y becomes $\pm\frac{1}{2}i$, which is excluded since y must be real. If we wished to permit y to become imaginary, our roots would be $\frac{3}{2} + \frac{1}{2}i\cdot i = \frac{3}{2} - \frac{1}{2} = 1$, and $\frac{3}{2} - \frac{1}{2}i\cdot i = 2$

These are the real roots of the given equation.

6. Method of Intersection of Curves in the Plane of the Complex Variable. (Rectangular Coordinates.) 54, 63, 75, 20.

This method has for it's main purpose the determination of a first approximation to the complex roots of an equation. The knowledge of such a first approximation is of great importance

in the methods of Newton, Schröder, Horner, and Weddle, as we shall see in Chapter IV.

In the given equation

$$f(z) = 0,$$

make a substitution as in the method of the preceding paragraph,

$$z = x + iy,$$

thus obtaining the two equations in x any y

$$u = 0, \ v = 0,$$

where u represents those terms of $f(x+iy)$ free of i, v those involving i. A value of $z = x+iy$ which is a root of $f(z) = 0$ must satisfy both equations $u = 0$ and $v = 0$ simultaneously; that is, the graphs of these two equations must intersect at that point.

In order to plot these graphs, proceed as follows: Substitute any constant value c_1 for x in the function u obtaining a function in y alone. Substitute for y successive positive integers $d_1, d_2, \ldots$ and determine the sign of the function for each value assigned. A change of sign between two successive integers will indicate that the curve $u = 0$ crosses the line $x = c_1$ between them. Repeat the process, using a new value $x = c_2$. It is well to use successive integers both positive and negative for $c_1, c_2, \ldots$. The intervals on the lines $x = c_1, x = c_2, \ldots$ should be plotted and approximate curves drawn through the positive and negative ends of the intervals. Since y enters only to even powers, the curve will be symmetric with respect to the x-axis. The true curve will not necessarily lie between these two curves, but from the nature of the plot, will in general lie wholly within the squares ~~areas bounded by the lines $x = c_i$ and $y = d_j$~~ through which the curves pass.

Having plotted approximately the curve $u = 0$, we should treat the curve $v = 0$ in a similar manner. Again we have symmetry with respect to the x-axis, for y enters only to the odd powers. This can be readily seen if we write $u = y\cdot\varphi(x,y^2)$. The φ-function is symmetric with respect to the x-axis. Hence $u = 0$ will be symmetric also, since the sign of the factor y cannot affect the vanishing of φ. For all values where $u \neq 0$, however, the function u will have opposite signs, for y taken positive, from the corresponding values for y negative.

The root of $f(z) = 0$ will lie in the squares through which the curves of both $u = 0$ and $v = 0$ may pass. If a smaller area is desired, it may be obtained by using fractional values for x and y. In this case only the area included in the squares in which the root lies needs to be examined.

Scheffler[63] has developed a similar method for finding the intersections of $u = 0$ and $v = 0$. He substitutes values of x and y as described above, but instead of determining merely the signs of the functions u and v, he computes their values at each point. These he plots, using the point (c_k, d_l) as origin, the value of u as the abscissa, that of v as the ordinate. In this way he obtains curves traced by the end-points of u and v, and thus finds more accurately the points where each curve $u = 0$ and $v = 0$ crosses the line $x = c_k$.

This method requires special care in the case of equal or nearly equal roots. If two roots were equal, the curves $u = 0$ and $v = 0$ would not cross each other at the point, but would merely become tangent. In this case one looking for roots might overlook the two. One is led to suspect the presence of roots

by the near approach of the curves to each other; in this case the interval in which this occurred should be investigated very carefully, or the equation examined for equal roots by one of the well-established methods. In fact, it is usually preferable to get rid of any rational or multiple roots before attempting to find complex roots.

This method is usually much simplified by the knowledge of upper and lower limits to the roots. Knowing these limits, one can confine himself to the intervals in which the roots lie and often avoid much tedious and fruitless labor in the parts of the plane lying outside of these boundaries.

Example: $z^4 + z^3 - 5z^2 - 8z + 10 = 0.$

$$u \equiv x^4 + x^3 - (6y^2 + 5)x^2 - (3y^2 + 8)x + y^4 + 5y^2 + 10 = 0.$$

$$v \equiv (4x^3 + 3x^2 - 4xy^2 - 10x - y^2 - 8)y = 0.$$

$u \equiv f(x, y^2)$

	$x = -5$	-4	-3	-2	-1	0	1	2	3	4	5
$y = 0$	+	+	+	+	+	+	−	−	+	+	+
1	+	+	+	+	+	+	−	−	−	+	+
2	−	−	−	−	+	+	−	−	−	−	+
3	−	−	−	−	+	+	+	−	−	−	−
4	−	−	−	+	+	+	+	−	−	−	−
5	−	−	−	+	+	+	+	−	−	−	−
6	−	−	−	+	+	+	+	+	−	−	−
7	−	−	+	+	+	+	+	+	−	−	−
8	−	−	+	+	+	+	+	+	+	−	−

$v \equiv y \cdot \varphi(x, y^2).$

	$x = -5$	-4	-3	-2	-1	0	1	2	3	4	5
$y = 0+$	+	+	+	+	−	−	−	+	+	+	+
1	+	+	+	+	−	−	−	+	+	+	+
2	+	+	+	−	−	−	−	−	+	+	+
3	+	+	−	−	−	−	−	−	−	+	+
4	+	−	−	−	−	−	−	−	−	−	+
5	−	−	−	−	−	−	−	−	−	−	−
6	−	−	−	−	−	−	−	−	−	−	−
7	−	−	−	−	−	−	−	−	−	−	−
8	−	−	−	−	−	−	−	−	−	−	−

These intervals as given in the tables are plotted on the following page, the area between the two v-curves being shown in black, that between the u-curves in red. Since the two intersect in the areas bounded by

$$x = -1,\ x = -3,\ y = 1,\ y = 2,$$

and

$$x = -1,\ x = -3,\ y = -1,\ y = -3,$$

we know that a complex root of f(z) lies in each of these intervals. Also, since the x-axis is part of the v-curve, the figure shows that real roots lie between 0 and 1, and between 2 and 3.

The approximations

$$x_1 + iy_1 \equiv -2 + i\cdot 1.5$$

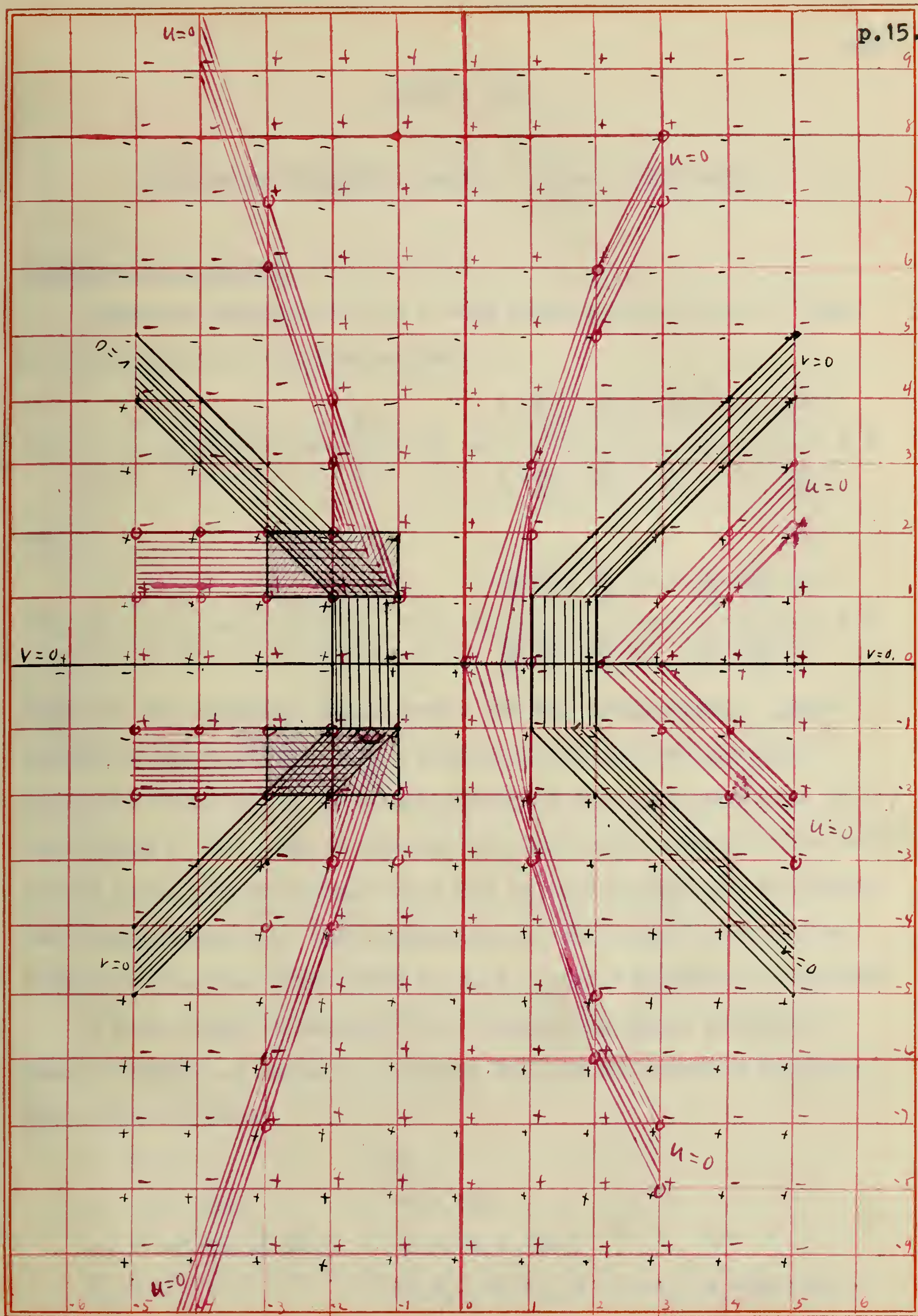
$$x_1 - iy_1 = -2 - i\cdot 1.5$$

will have real parts correct to within 1 unit, and imaginary parts correct to within 0.5 unit.

The solution of this equation by Horner's method, given on page 41, finds the roots to be approximately

$$x + iy \equiv -2.009(15) + 1.081(99)i,$$

$$x - iy \equiv -2.009(15) - 1.081(99)i.$$


u=0
u=0
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CHAPTER III.

METHODS OF REPEATED LIMITS. (POLAR COORDINATES).

7. Bernoulli's Method. [2, 24, 28, 31, 74]

Given an equation $f(x) = 0$ with roots $x_1, x_2, x_3, \ldots x_n$, where $|x_1| \geqq |x_2| \geqq |x_3| \geqq \ldots \geqq |x_n|$; then we have

$$(1)\quad \underset{m \doteq \infty}{L} \frac{x_1^m + x_2^m + x_3^m + \ldots + x_n^m}{x_1^{m-1} + x_2^{m-1} + x_3^{m-1} + \ldots + x_n^{m-1}} = \underset{m \doteq \infty}{L}\, x_1 \frac{1 + \left(\frac{x_2}{x_1}\right)^m + \left(\frac{x_3}{x_1}\right)^m + \left(\frac{x_4}{x_1}\right)^m + \cdots + \left(\frac{x_n}{x_1}\right)^m}{1 + \left(\frac{x_2}{x_1}\right)^{m-1} + \left(\frac{x_3}{x_1}\right)^{m-1} + \left(\frac{x_4}{x_1}\right)^{m-1} + \cdots + \left(\frac{x_n}{x_1}\right)^{m-1}} = x_1.$$

and

$$(2)\quad \underset{m \doteq \infty}{L} \frac{x_1^{-m} + x_2^{-m} + x_3^{-m} + \ldots + x_n^{-m}}{x_1^{-m-1} + x_2^{-m-1} + x_3^{-m-1} + \ldots + x_n^{-m-1}} = \underset{m \doteq \infty}{L}\, x_n \frac{\left(\frac{x_1}{x_n}\right)^{-m} + \left(\frac{x_2}{x_n}\right)^{-m} + \left(\frac{x_3}{x_n}\right)^{-m} + \cdots + \left(\frac{x_{n-2}}{x_n}\right)^{-m} + \left(\frac{x_{n-1}}{x_n}\right)^{-m} + 1}{\left(\frac{x_1}{x_n}\right)^{-m-1} + \left(\frac{x_2}{x_n}\right)^{-m-1} + \left(\frac{x_3}{x_n}\right)^{-m-1} + \cdots + \left(\frac{x_{n-2}}{x_n}\right)^{-m-1} + \left(\frac{x_{n-1}}{x_n}\right)^{-m-1} + 1} = x_n.$$

Build up the equations whose roots are the 1st, 2nd, 3rd, , , , mth powers of the roots of $f(x) = 0$ and from the $(m-1)$th and (m)th equations calculate x by substituting in the left-hand side of (1) the values of the sums of the $(m-1)$th and (m)th powers of the roots of the given equation. These sums can be easily obtained by taking the negative ratio of the coefficient of the second term to the coefficient of the first term in each of the respective equations.

A more powerful method [56] for determining these successive sums of powers of roots is given by the use of Newton's Formulae. From these we have:

$$\begin{aligned}
s_1 &= -a_1, &&= -a_1 \\
s_2 &= a_1^2 - 2a_2, &&= -a_1 s_1 - 2a_2 \\
s_3 &= -a_1^3 + 3a_1 a_2 - 3a_3, &&= -a_1 s_2 - a_2 s_1 - 3a_3 \\
&\ldots \\
s_{n-1} &= &&-a_1 s_{n-2} - a_2 s_{n-3} - \ldots - a_{n-2} s_1 - (n-1)a_{n-1}.
\end{aligned}$$

where $s_j = x_1^j + x_2^j + x_3^j + \dots + x_n^j$, and where $a_1 \dots a_n$ are the coefficients of the terms of $f(x) = 0$, when a_0, the coefficient of the first term is 1. For values of $s_n, s_{n+1}, \dots$ we have the formula

$$s_m + a_1 s_{m-1} + a_2 s_{m-2} + \dots + a_n s_{m-n} = 0.$$

By means of these formulae we can calculate s_m for values of m as large as we desire.

In case the equation $f(x) = 0$ has two or more real roots of equal largest absolute value the ratio $\frac{s_m}{s_{m-1}}$ may not give the value of one of the roots. If, say, of the $\mu + \nu$ roots of equal, largest absolute value, μ ~~of them~~ are positive and ν negative, we have [56]

$$\underset{m \doteq \infty}{L} \frac{s_{2m+2}}{s_{2m+1}} = \frac{\mu + \nu}{\mu - \nu} |x_1|$$

and

$$\underset{m \doteq \infty}{L} \frac{s_{2m+1}}{s_{2m}} = \frac{\mu - \nu}{\mu + \nu} |x_1|$$

The ratio $\frac{s_{m+1}}{s_m}$ would not approach a limit therefore. However,

$$\underset{m \doteq \infty}{L} \frac{s_{2m+2}}{s_{2m}} = \underset{m \doteq \infty}{L} \frac{s_{2m+1}}{s_{2m-1}} = x_1^2 = x_2^2 = \dots = x_{\mu+\nu}^2.$$

In case $\mu = \nu$, $\underset{m \doteq \infty}{L} \frac{s_{2m+1}}{s_{2m-1}}$ becomes indeterminate, but $\underset{m \doteq \infty}{L} \frac{s_{2m+2}}{s_{2m}}$ still gives the desired values.

If the two roots of largest absolute value are conjugate complex, say $\rho(\cos\theta + i \sin\theta)$ and $\rho(\cos\theta - i\sin\theta)$, it is possible to determine first the angle θ. We may write:

$$s_m = 2\rho^m \cos m\theta + (x_3^m + x_4^m + x_5^m + x_6^m + \dots + x_n^m)$$

$$= 2\rho^m \cos m\theta + c_m(n - 2)\rho_3^m,$$

where ρ_3 is the absolute value of x_3, which is less than ρ; and c_m is a constant less than 1. The ratio $\frac{s_{m+1}}{s_m}$ will not approach

a limit;for we have

$$\frac{s_{m+1}}{s_m} = \frac{2\rho^{m+1}\cos(m+1)\theta + c_{m+1}(n-2)\rho_3^{m+1}}{2\rho^m\cos(m)\theta + c_m(n-2)\rho_3^m}$$

Here the cosine functions in both numerator and denominator, separated by the phase θ at all times,can each take values small enough to make the term in which they appear too small for the second term to be neglected,no matter how large m be taken.Moreover since each oscillates between 1 and -1,both numerator and denominator will take positive and negative values, which may for certain values of θ cause the ratio to have different signs and therefore not approach a definite limit.

If the sequence[56]

(3) sgn cos θ ,sgn cos 2θ ,.... sgn cos $n\theta$,.....

is known,we can approximate to θ in the following manner:

Let ν_1 be the number of positive terms before the first negative term of the sequence ;let ν_2 be the number of successive negative terms before the next positive term,and so forth.Then, since we may assume without loss of generality that $\theta < \pi$,we know:

$\nu_1\theta < \frac{\pi}{2} < (\nu_1+1)\theta$ or, $\frac{\pi}{2(\nu_1+1)} < \theta < \frac{\pi}{2\nu_1}$.

$(\nu_1+\nu_2)\theta < \frac{3\pi}{2} < (\nu_1+\nu_2+1)\theta$ $\qquad \frac{3\pi}{2(\nu_1+\nu_2+1)} < \theta < \frac{3\pi}{2(\nu_1+\nu_2)}$

$(\nu_1+\nu_2+\nu_3)\theta < \frac{5\pi}{2} < (\nu_1+\nu_2+\nu_3+1)\theta$ $\qquad \frac{5\pi}{2(\nu_1+\nu_2+\nu_3+1)} < \theta < \frac{5\pi}{2(\nu_1+\nu_2+\nu_3)}$

.... $\qquad$ ....

from which

(4) $$\theta = \underset{\lambda=\infty}{L} \frac{(2\lambda-1)\pi}{2(\nu_1+\nu_2+\cdots+\nu_\lambda)} = \underset{\lambda=\infty}{L} \frac{\lambda\pi}{(\nu_1+\nu_2+\cdots+\nu_\lambda)}.$$

More briefly,if w_m is the number of changes of sign in the first m terms of the sequence (3),we have

$$\theta = \underset{m=\infty}{L} \frac{w_m\pi}{m}.$$

The sequence

$$\text{sgn } s_1, \text{sgn } s_2, \ldots\ldots \text{sgn } s_m, \ldots\ldots$$

may agree with the sequence (3) in such a manner that θ can be determined from it by the use of (4). If however $\frac{\rho}{\rho_3}$ is not sufficiently small, this is not possible. Netto ([56] p.267) gives the conditions for this as being dependent on ρ_1, ρ_3, and c_m, which are unknown; so that the conditions are valueless in practical use.

Netto[56] attributes the following modification to Jacobi, who followed a suggestion by Fourier:

Given an equation which has μ roots of equal largest absolute values with the absolute values of the sums of the (k)th, (k+1)th, ... powers of the roots $x_1, x_2, x_3, \ldots x_\mu$ sufficiently large in comparison with the sums of corresponding powers of the remaining roots so that we may neglect powers of those roots smaller than x_μ, and write:

$$s_k = x_1^k + x_2^k + \ldots\ldots + x_\mu^k$$
$$s_{k+1} = x_1^{k+1} + x_2^{k+1} + \ldots\ldots + x_\mu^{k+1}$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

We know that the unknown coefficients of the equation

(5) $$x^\mu + \gamma_1 x^{\mu-1} + \gamma_2 x^{\mu-2} + \ldots\ldots + \gamma_\mu = 0$$

with roots $x_1, x_2, \ldots x_\mu$ can be determined from the relations used above, namely:

(6) $$s_{k+\mu} + \gamma_1 s_{k+\mu-1} + \ldots\ldots + \gamma_\mu s_k = 0$$
$$s_{k+\mu+1} + \gamma_1 s_{k+\mu} + \ldots\ldots + \gamma_\mu s_{k+1} = 0$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$s_{k+2\mu-1} + \gamma_1 s_{k+2\mu-2} + \ldots\ldots + \gamma_\mu s_{k+\mu-1} = 0.$$

Eliminating $\gamma_1, \gamma_2 \cdots \gamma_\mu$, from equation (5) and the set of equations numbered (6), we have

$$\begin{vmatrix} x^{\mu} & x^{\mu-1} & . & . & . & 1 \\ s_{k+\mu} & s_{k+\mu-1} & . & . & . & s_k \\ . & . & . & . & . & . \\ s_{k+2\mu-1} & s_{k+2\mu-2} & . & . & . & s_{k+\mu-1} \end{vmatrix} = 0.$$

For $\mu = 2$, as is the case when we have two conjugate complex roots, this determinant reduces to the following:

(7) $(s_{k+1}^2 - s_k s_{k+2})x^2 - (s_{k+1}\, s_{k+2} - s_k s_{k+3})x + (s_{k+2}^2 - s_{k+1}\, s_{k+3}) = 0.$

Laisant [40] gives a sequence for determining roots of equations which is more general than the sequence made up of the sums of the successive powers of the roots. He uses Newton's formula:

$$s_m + a_1 s_{m-1} + a_2 s_{m-2} + \dots + a_n s_{m-n} = 0,$$

and proves that if a sequence has for a law of recurrence

$$u_m + a_1 u_{m-1} + a_2 u_{m-2} + \dots + a_n u_{m-n} = 0,$$

we may choose any arbitrary values $u_{m-1}, u_{m-2}, \dots u_{m-n}$, and the sequence built from these can be used for determining the roots of the equation with coefficients $1, a_1, a_2, \dots a_n$, in the same manner as the sequence made up of the sums of successive powers of the roots.

Example: Find the complex roots of the equation

$$x^3 - 3x^2 + x + 5 = 0.$$

By the use of Newton's formulae we get the following sequence:

3, 3, 7, 3, -13, -77, -233, -557, -1053, -1431, -455, 5331, 23693, 67753, 153001, 273235, ...

Selecting the last four terms as s_k, s_{k+1}, s_{k+2}, and s_{k+3} respectively, and substituting in the formula (7) above, we obtain the following equation:

$$9792x^2 - 38729x + 47690 = 0.$$

The roots of this equation are 1.977 + 0.986i, 1.977 - 0.986i. The correct values of the roots of the original equation are 2 + i, 2 - i. This is a very special equation, as the real root is 1, which makes the sums of powers of the complex roots grow large very rapidly in comparison. In general this is not the case, so that a larger number of terms of the sequence will need to be found to obtain equally accurate results.

8. Gräffe's Method. 1, 10, 20, 28, 31, 56, 72.

The method proposed by D. Bernoulli gave* only the roots of greatest and least absolute value. In 1837 C.H. Gräffe[31] developed a method which used many of the same principles involved in Bernoulli's method, but which gave all the roots of an equation at once, with a more rapid approach to the result. His method is as follows:

Given the equation $f(x) = 0$ with roots $x_1, x_2, \ldots\ x_n$. The product $f(x).f(-x) = 0$ will be an equation of the $(2n)$th degree with roots $x_1, -x_1, x_2, -x_2, \ldots\ x_n, -x_n$. In performing the multiplication all odd powers of x will disappear, and the substitution

$$x' = x^2$$

will therefore give an equation of the (n)th degree in x',

$$(x' - x_1^2)(x' - x_2^2)\ldots(x' - x_n^2) = 0,$$

whose roots $x_1^2, x_2^2, \ldots\ x_n^2$ are the squares of the roots of the original equation.

The new equation can be operated on as was the original equation, and the process continued indefinitely, giving equations

*Meyer extended it to all roots later. Math. Annalen, v.33, 1889. p.511.

with roots

$$x_1^4, x_2^4, x_3^4, \ldots\ldots x_n^4$$
$$x_1^8, x_2^8, x_3^8, \ldots\ldots x_n^8$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$x_1^{2^m}, x_2^{2^m}, x_3^{2^m}, \ldots\ldots x_n^{2^m}$$

Gräffe then proves that if all the roots are real and unequal so that

$$|x_1| > |x_2| > |x_3| > \ldots\ldots > |x_n|$$

the equation of the (r)th powers of the roots takes the form (approximately),

$$(1) \qquad x^n - x_1^r x^{n-1} + x_1^r x_2^r x^{n-2} - x_1^r x_2^r x_3^r x^{n-3} + \ldots\ldots \pm x_1^r x_2^r x_3^r \ldots x_n^r = 0,$$

for all values of r sufficiently large. His proof is based on the symmetric functions of the roots of an equation whose coefficient of the term of highest degree in x is unity.

$$\text{Coefficient of } x^{n-1} = -(x_1^r + x_2^r + x_3^r + \ldots + x_n^r)$$
$$\text{Coefficient of } x^{n-2} = (x_1^r x_2^r + x_1^r x_3^r + \ldots + x_1^r x_n^r + x_2^r x_3^r + \ldots + x_{n-1}^r x_n^r).$$
$$\text{Coefficient of } x^{n-3} = -(x_1^r x_2^r x_3^r + x_1^r x_2^r x_4^r \ldots + x_1^r x_2^r x_n^r + \ldots + x_{n-2}^r x_{n-1}^r x_n^r)$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$\text{Coefficient of } x^0 = (-1)^n x_1^r x_2^r x_3^r \ldots x_n^r.$$

In each case the first term, involving powers of roots large in comparison to those of the following terms, would for increasing r become so large that the remaining terms could be neglected, thus giving equation (1) above.

From equation (1) any real root can be found. For example, if x_k is desired, it can be determined from the following ratio:

$$x_k^r = -\frac{\text{coefficient of } x^{n-k}}{\text{coefficient of } x^{n-k-1}} = \frac{x_1^r x_2^r x_3^r \ldots x_{k-1}^r x_k^r}{x_1^r x_2^r x_3^r \ldots x_{k-1}^r}$$

Now consider the case where the two roots of largest absolute

value are conjugate complex. Since they have equal absolute value, the assumption previously made that the absolute value of x_1 is greater than the absolute value of x_2 no longer holds. We may assume the following, however:

$$|x_1| = |x_2| > |x_3| > \ldots\ldots > |x_n| .$$

Equation (1) now becomes:

$$(2)\quad x^n - (x_1^n + x_2^n)x^{n-1} + x_1^n x_2^n x^{n-2} - x_1^n x_2^n x_3^n x^{n-3} + \ldots\ldots \pm x_1^n x_2^n \ldots x_n^n = 0.$$

In this case the value of x_1 can be determined* from the relation

$$|x_1^n| = |x_2^n| = \left|\sqrt{x_1^n x_2^n}\right|$$

In case the complex roots are not of greatest absolute value, the necessary assumption becomes:

$$|x_1| > |x_2| > \ldots > |x_k| = |x_{k+1}| > |x_{k+2}| > \ldots > |x_n| .$$

Equation (1) then becomes as follows:

$$x^n - x_1^n x^{n-1} + \ldots + (-1)^{k-1}\cdot(x_1^n \ldots x_{k-1}^n)x^{n-k+1}$$
$$+ (-1)^k (x_1^n x_2^n \ldots x_{k-1}^n x_k^n + x_1^n x_2^n \ldots x_{k-1}^n x_{k+1}^n)x^{n-k} + (-1)^{k+1}(x_1^n x_2^n \ldots x_k^n x_{k+1}^n)x^{n-k-1}$$
$$\pm \ldots\ldots \pm x_1^n x_2^n \ldots x_{n-1}^n x_n^n = 0.$$

The absolute value of x_k and x_{k+1} can be obtained from the coefficients of x^{n-k+1}, x^{n-k}, and x^{n-k-1}.

To determine the argument of the roots x_1 and x_2, we make use of the fact that $\sum_{j=1}^{n} x_j = -a_1$ (assuming that $a_0 = 1$.) If we determine first the absolute values of all the roots, we can find the real roots by substitution in $f(x) = 0$ to determine the sign of each. Then, representing the two complex roots by $\rho_1(\cos\theta_1 + i\sin\theta_1)$ and $\rho_1(\cos\theta_1 - i\sin\theta_1)$, we have the relation:

$$(3)\qquad 2\rho_1\cos\theta_1 + x_3 + x_4 + \ldots + x_n = -a_1,$$

or,

$$\cos\theta_1 = \frac{-a_1 - x_3 - x_4 - \ldots - x_n}{2\rho_1}$$

*$|x_1|$ and $|x_2|$ cannot be determined from the coefficient $(x_1^n + x_2^n)$ since $|x_1^n| + |x_2^n|$ may be greater than $|x_1^n + x_2^n|$.

If the equation has two pairs of complex roots, the corresponding to equation (3) will contain two unknowns $\cos\theta_1$ and $\cos\theta_2$. In this case the following additional condition is needed:

$$\frac{2\cdot\cos\theta_1}{\rho_1}+\frac{2\cdot\cos\theta_2}{\rho_2}+\frac{1}{x_5}+\frac{1}{x_6}+\cdots\cdots+\frac{1}{x_n}=\frac{-a_n}{a_{n-1}}$$

This will give both values of θ when combined with (3) which now becomes as follows:

$$2\rho_1\cos\theta_1+2\rho_2\cos\theta_2+x_5+x_6+\cdots+x_n=-a_1 \quad .$$

If more than two pairs of complex roots occur, the following method[60] will determine the arguments of the roots:

Let the degree of the equation be even, equal to 2m. (If it is odd, multiply by x throughout, making it even). If the equation is divided by x^m, it will take the following form:

$$a_0x^m+a_1x^{m-1}\cdots+a_{2m}x^{-m}=0.$$

If now $\rho e^{i\theta}$ be substituted for x and the real and imaginary parts each equated to zero, the two equations become as follows:

$$a_0'\cos m\theta+a_1'\cos(m-1)\theta+\cdots+a_m'=0$$

$$b_0'\sin m\theta+b_1'\sin(m-1)\theta+\cdots+b_{m-1}'\sin\theta=0,$$

where

$$a_0'=a_0\rho^m+a_{2m}\rho^{-m}=b_0'$$

$$a_1'=a_1\rho^{m-1}+a_{2m-1}\rho^{-m+1}=b_1'$$

$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$

$$a_{m-1}'=a_{m-1}\rho^{+1}+a_{m+1}\rho^{-1}=b_{m-1}'$$

$$a_m'=a_m$$

From Trigonometry we get the following formulae:

$$\cos 2\theta=2\cos^2\theta-1,$$

$$\cos 3\theta=4\cos^3\theta-3\cos\theta$$

$$\cos 4\theta=8\cos^4\theta-8\cos^2\theta+1,$$

$$\cos 5\theta=16\cos^5\theta-20\cos^3\theta+5\cos\theta$$

$$\frac{\sin 2\theta}{\sin\theta} = 2\cos\theta$$

$$\frac{\sin 3\theta}{\sin\theta} = 4\cos^2\theta - 1.$$

$$\frac{\sin 4\theta}{\sin\theta} = 8\cos^3\theta - 4\cos\theta$$

$$\frac{\sin 5\theta}{\sin\theta} = 16\cos^4\theta - 12\cos^2\theta + 1.$$

These values substituted in the two equations above will give for each absolute value ρ of a root two equations in ~~$\sin\theta$ and~~ $\cos\theta$ from which θ can be calculated, (as in par. 4, p.5, by finding the greatest common divisor).

For the case in which more than two roots have equal absolute value a method corresponding to the case of two roots of equal absolute value has been given. C. Runge [60] gave in 1900 the following method which deals with a more general case:

Given an equation $f(x) \equiv a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_n = 0$. If the roots of this equation are such that $x_1, x_2, \ldots x_\mu$ are large in absolute value in comparison with $x_{\mu+1}, x_{\mu+2}, \ldots x_n$, the given equation may be broken up into two equations:

$$a_0x^\mu + a_1x^{\mu-1} + a_2x^{\mu-2} + \ldots + a_\mu = 0,$$

$$a_\mu x^{n-\mu} + a_{\mu-1}x^{n-\mu+1} + \ldots + a_n = 0,$$

such that the first equation will have the roots $x_1, x_2, \ldots x_\mu$, approximately, and the latter the roots $x_{\mu+1}, x_{\mu+2}, \ldots x_n$, approximately. If, in turn, the roots of these equations can be divided into sets one of which contains only roots large in somparison with the roots of the other set, the two equations can again be broken up, and the process continued until the resulting equations become of as low a degree as is desired, or until all the roots of any one equation become of absolute value such that no one is

sufficiently large in comparison with the others to permit of further breaking up of the equation; i.e., all roots have equal absolute value.

Conversely, if each of the coefficients $a_0, a_1, \dots a_\mu$ is large in comparison with each of $a_{\mu+1}, a_{\mu+2}, \dots a_n$, the equation

$$f(x) \equiv a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \dots + a_n = 0$$

can be broken up into two equations

$$a_0 x^\mu + a_1 x^{\mu-1} + a_2 x^{\mu-2} + a_3 x^{\mu-3} + \dots + a_\mu = 0,$$

$$a_\mu \quad x^{n-\mu} + a_{\mu+1} x^{n-\mu+1} + \dots + a_n = 0,$$

the first of which will give approximately the μ roots of largest absolute value of $f(x) = 0$, while the second will give the remaining $n-\mu$ roots.

This gives a means of simplifying the solution of any equation with several real or complex roots of approximately equal absolute value; for if the equation of the powers of the roots of the given equation is built up as previously outlined, the equation will separate into equations of lower degree which can be solved more readily than the original.

To simplify the method of obtaining equations each with roots equal to the square of the roots of the preceding equation, we have the following:

$$f(x) \equiv a_0 x^n + a_1 x^{n-1} + \dots + a_n = 0$$

$$f(-x) \equiv a_0 x^n - a_1 x^{n-1} + \dots \pm a_n = 0.$$

Then $f(x) \cdot f(-x) \equiv a_0 x^{2n} - (a_1^2 - 2a_0 a_2) x^{2n-2} + (a_2^2 - 2a_1 a_3 + 2a_0 a_4) x^{2n-4} + \dots$

$$+ (-1)^K \quad (a_K^2 - 2a_{K-1} a_{K+1} + 2a_{K-2} a_{K+2} - 2a_{K-3} a_{K+3} + \dots \pm 2a_0 a_{2K}) x \pm \dots$$

$$+ (-1)^n a_n^2 .$$

or, $\quad b_0 x'^n + b_1 x'^{n-1} + b_2 x'^{n-2} + \dots + b_n = 0,$

where $\quad b_0 = a_0^2$

$$b_1 = -(a_1^2 - 2a_0 a_2)$$

$$b_2 = (a_2^2 - 2a_1a_3 + 2a_0a_4)$$

$$b_3 = -(a_3^2 - 2\,a_2a_4 + 2\,a_1a_5 - 2\,a_0a_6)$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$b_k = (-1)^k(a_k^2 - 2a_{k-1}a_{k+1} + 2a_{k-2}a_{k+2} - 2a_{k-3}a_{k+3} \cdots \pm 2a_0\,a_{2k})$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$b_n = (-1)^n \cdot a_n^2$$

With these relations the step from one equation to the next can easily be made.

Example: Find all the roote of the equation:

$$x^4 + x^3 - 5x^2 - 7x + 10 = 0.$$

Power of roots.	a_0	a_1	a_2	a_3	a_4
1	1	1	−5	−7	10
	$a_0^2=1$	$a_1^2=1$	$a_2^2=25$	$a_3^2=49$	$a_4^2=100$
		$a_0a_2=10$	$a_1a_3=14$	$a_2a_4=100$	
			$a_0a_4=20$		
2^1	1	11	59	149	100
	1	121	3481	222_2 *	100_2 *
		−118	−3278	-118_2	
			200		
2^2	1	3	403	104_2	100_2
	1	9	162_3	108_6	100_6
		−806	62_3	8_6	
			20_3		
2^3	1	−797	120_3	100_6	100_6

* 222_2 signifies $222\cdot10^2$; 100_2, $100\cdot10^2$; 108_6, $108\cdot10^6$; etc.

Power of roots	a_0	a_1	a_2	a_3	a_4
	1	635_3	144_8	100_{14}	100_{14}
		-240_3	1594_8	0_{14}	
			2_8		
2^4	1	395_3	1740_8	100_{14}	100_{14}
	1	156_9	303_{20}	100_{30}	100_{30}
		-348_9	-79_{20}	0_{30}	
			0_{20}		
2^5	1	-192_9	226_{20}	100_{30}	100_{30}
	1	368_{20}	511_{42}	100_{62}	100_{62}
		-448_{20}	38_{42}	0	
			0_{42}		
2^6	1	-80_{20}	549_{42}	100_{62}	100_{62}
	1	64_{42}	3014_{86}	100_{126}	100_{126}
		-1098_{42}	2_{86}	0_{126}	
			0_{86}		
2^7	1	-1034_{42}	3416_{86}	100_{126}	100_{126}
	1	107_{88}	912_{174}	100_{254}	100_{254}
		60_{88}	0_{174}	0_{174}	

To carry the work beyond the 2^7 power of the roots is useless, as four of the coefficients are not affected by further transformations, while the second coefficient will continue to oscillate indefinitely. Using the equation of the 2^7 power of the roots, we have the following:

$$x'^4 + 1034_{42} x'^3 + 3416_{86} x'^2 + 100_{126} x' + 100_{126} = 0. \quad x' = x$$

$$x_4 = \sqrt[2^7]{\frac{100_{126}}{100_{126}}} = 1.$$

$$x_3 = \sqrt[2^7]{\frac{100_{126}}{3016_{86}}} = 1.9996$$

By substitution in $f(x) = 0$ we find that both these roots are positive.

$$\rho_1 = \rho_2 = \sqrt[256]{3016_{86}} = 2.2358$$

$$x_1 + x_2 + x_3 + x_4 = 1 + 1.9996 + 2(2.2358)\cos\theta = -1.$$

$$\cos\theta = -\frac{3.9996}{4.4716}$$

$$\theta = 153^\circ\, 26'\, 20''.$$

$$x_1 = 2.2358(\cos 153^\circ 26'20'' + i \sin 153^\circ 26'20''$$

$$= -1.9998 + 0.9998i$$

$$x_2 = -1.9998 - 0.9998i$$

(The correct values of the roots are 1, 2, $-2+i$, $-2-i$.)

9. Hayashi's Method.

T. Hayashi[34] has given a method for the determination of complex roots, which he applies to an equation of the third degree; but the method is readily extended, as he remarks, to ~~any~~ equations of any degree. For the cubic, the method is as follows:

Given the cubic equation

$$a_0 x^3 + a_1 x^2 + a_2 x + a_3 = 0,$$

with real coefficients; and let the two complex roots be $\rho e^{i\theta}$ and $\rho e^{-i\theta}$. If we write ε for $e^{i\theta}$ and b_0, b_1, b_2, b_3, for $a_0\rho^3, a_1\rho^2, a_2\rho, a_3$ respectively, we have the following:

$$b_0\varepsilon^3 + b_1\varepsilon^2 + b_2\varepsilon + b_3 = 0,$$

$$b_0\varepsilon^{-3} + b_1\varepsilon^{-2} + b_2\varepsilon^{-1} + b_3 = 0.$$

Multiplying the latter by ε^3 we obtain

$$b_3\varepsilon^3+b_2\varepsilon^2+b_1\varepsilon+b_0 = 0$$

Therefore ε is a common root of

$$b_0x^3+b_1x^2+b_2x+b_3= 0,$$

and $$b_3x^3+b_2x^2+b_1x+b_0= 0.$$

Combining to get rid of the terms in x^3, we have

$$(b_1b_3-b_2b_0)x^2+(b_2b_3-b_1b_0)x+(b_3^2-b_0^2) = 0,$$

of which ε is a root. But ε^{-1} is also a root. Therefore,

(1) $$(b_1b_3-b_2b_0)\varepsilon^2+(b_2b_3-b_1b_0)\varepsilon+(b_3^2-b_0^2) = 0,$$

$$(b_1b_3-b_2b_0)\varepsilon^{-2}+(b_2b_3-b_1b_0)\varepsilon^{-1}+(b_3^2-b_0^2) = 0.$$

Multiplying the latter by ε^2,

(2) $$(b_3^2-b_0^2)\varepsilon^2+(b_2b_3-b_1b_0)\varepsilon+(b_1b_3-b_2b_0) = 0.$$

Subtracting (1) from (2), and factoring,

$$[(b_3^2-b_0^2)-(b_1b_3-b_2b_0)](\varepsilon^2-1) = 0$$

Since for complex roots $\theta \neq 0$ and therefore $(\varepsilon^2-1)\neq 0$, we have

$$(b_3^2-b_0^2)-(b_1b_3-b_2b_0) = 0.$$

Replacing the letters b_0, b_1, b_2, b_3 by their respective values, this equation becomes

$$a_0^2\rho^6-a_0a_2\rho^4+a_1a_3\rho^2-a_3^2 = 0.$$

This is a cubic equation in ρ^2 and can be solved for real, positive roots, among which will be found the absolute value ρ of the roots $\rho e^{i\theta}$, and $\rho e^{-i\theta}$.

The method of procedure in the general case from this point on is not clearly indicated. No means is given for determining which real positive root of the last equation will be the absolute value of a root of the original equation. In the equation of the third degree this equation in ρ^2 is the equation whose roots are equal

to the products of pairs of roots of the original equation, which can have only one real root, namely, the product of the two complex roots of the given equation. For the equation of the fourth degree, however, the equation in ρ^2 becomes of degree 8; for the fifth degree equation, of degree 20; for the (n)th degree equation, of degree $n.2^{n-3}$; so that obviously the roots are no longer the product of pairs of roots of the original equation. No statement is made as to the possibility of positive roots entering the equation in ρ^2 which do not give absolute values of complex roots of the given equation.

Another disadvantage of the method is the rapid increase of the degree of the equation in ρ^2 for increasing n. For example, to solve an equation of the sixth degree it is necessary to find the real positive roots of an equation of degree 48. In this respect the method offers little advantage over the methods of Lagrange * and Taylor #; for while the equation to be solved for real roots can be more readily built up than by those two methods, it is in general of so much higher degree that any advantage of simplicity of formation is overcome by the disadvantage of a longer and more difficult solution.

Example; $\quad x^4+x^3-5x^2-7x+10 = 0.$

We treat first the general equation of the fourth degree as in the case of the cubic discussed above. Placing $e^{i\theta} = \varepsilon$ and b_0, b_1, b_2, b_3, b_4, equal to $\rho^4 a_0, \rho^3 a_1, \rho^2 a_2, \rho a_3, a_4$ respectively, and reducing to an equation of the second degree as before we get the equation

$$(b_2b_4^3 - b_0b_2b_4^2 - b_0^2b_2b_4 + b_0^3b_2 - b_1b_3b_4^2 + b_0b_1^2b_4 + b_0b_3^2b_4 - b_0^2b_1b_3)$$
$$+(b_3b_4^3 - b_0b_1b_4^2 - b_0^2b_3b_4 + b_0^3b_1 - b_1b_2b_4^2 + b_0b_1b_2b_4 + b_0b_2b_3b_4 - b_0^2b_2b_3)$$
$$+(b_4^4 - 2b_0^2b_4^2 + b_0^4 - b_1^2b_4^2 + 2b_0b_1b_3b_4 - b_0^2b_3^2) = 0.$$

*Cf. p. 7. #Cf. p. 9.

Eliminating the second term ~~as before~~ and replacing b_j by it's value in terms of a_j and ρ we obtain the following equation:

$$(a_0^4)\rho^{16} - (a_0^3 a_2)\rho^{14} + (a_0^2 a_1 a_3)\rho^{12} + (a_0^2 a_2 a_4 - a_0 a_1^2 a_4 - a_0^2 a_3^2)\rho^{10}$$
$$+ (2a_0 a_1 a_3 a_4 - 2a_0^2 a_4^2)\rho^8 + (a_0 a_2 a_4^2 \quad - a_0 a_3^2 a_4 - a_1^2 a_4^2)\rho^6 + (a_1 a_3 a_4^2)\rho^4$$
$$- (a_2 a_4^3)\rho^2 + a_4^4 = 0.$$

In the given equation, $a_0 = 1, a_1 = 1, a_2 = -5, a_3 = -7, a_4 = 10.$ Making this substitution we get from the equation above:

$$\rho^{16} + 5\rho^{14} - 7\rho^{12} - 109\rho^{10} - 340\rho^8 - 1090\rho^6 - 700\rho^4 + 5000\rho^2 + 10000 = 0.$$

The real positive roots of this equation are $\sqrt{5}$ and $\sqrt{2}$. There are two conjugate complex roots of absolute value $\sqrt{5}$. The arguments of these roots can be found from the quadratic equation in ε given above. There are no complex roots with absolute value $\sqrt{2}$.

10. Solution by Means of the Equation of Products of Pairs of Roots.

The following method was suggested by the method of Hayashi when applied to equations of the third degree. Hayashi forms an equation from the given equation by means of the process described in the preceding article. In the case that the given equation is a cubic, the new equation turns out to be that equation whose roots are the products of the roots of the original equation taken together in pairs. For higher degrees this is not the case; but the lower degree equation suggested that there might be some value in a general method which gave such results for all degrees.

Thus, given an equation of degree n with roots $x_1 \ldots . x_n$, form by means of symmetric functions of the roots the equation whose roots are

$$x_1 x_2,\ x_1 x_3,\ \ldots .\ x_1 x_n,$$

$$x_2x_3, \ldots\ldots x_2x_n,$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$x_{n-1}x_n.$$

This equation will be of degree $\frac{n(n-1)}{2}$. Since the product of two conjugate complex quantities is always a real positive quantity, we need only solve this new equation for real positive roots. From these can be found the absolute values of each of the complex roots of the original equation. For example, if x_1 and x_2 are conjugate complex, we have the relation:

$$|x_1| = |x_2| = \left|\sqrt{x_1x_2}\right|$$

For the equation of the fourth degree

$$a_0x^4 + a_1x^3 + a_2x^2 + a_3x + a_4 = 0$$

with roots x_1, x_2, x_3, x_4, the new equation becomes as follows:

$$a_0x'^6 - a_0^2a_2x'^5 + a_0^2(a_1a_3 - a_4)x'^4 - (a_1^2a_4 - 2a_0a_2a_4 + a_0a_3^2)x'^3$$
$$+ a_0a_4(a_1a_3 - a_4)x'^2 - a_2a_4^2x' + a_4^3 = 0$$

whose roots are $x_1x_2, x_1x_3, x_1x_4, x_2x_3, x_2x_4, x_3x_4$.

This method has one disadvantage of the method of Hayashi in that each positive root of the new equation may not give the absolute value of a pair of complex roots of the original equation. It has the advantage of giving an equation of lower degree to be solved for real roots, however; while the knowledge that the roots of the new equation are going to be the products of pairs of roots of the original equation may make it possible to eliminate at least in some cases the disadvantage mentioned above.

Example: Find the complex roots of the equation

$$x^4 + 5x^3 + 7x^2 - 3x - 10 = 0.$$

The equation of the products of pairs of roots is the following:

$x'^6 - 7x'^5 - 5x'^4 + 101x'^3 + 50x'^2 - 700x' - 1000 = 0.$

The roots of this equation are 5,-2,and four imaginary roots.Since four are imaginary,the original equation must have at least two complex roots. Since 5 is the only real positive root,it cannot have more than one pair of conjugate complex roots.Therefore the original equation has a pair of complex roots and two real roots. The absolute value of each of the complex roots is $\sqrt{5}$.

11.Method of Intersection of Curves in the Plane of the Complex Variable.(Polar Coordinates).[63,54,75,20]

The method as outlined in paragraph 6 can also be carried out by means of polar coordinates.Given the equation

$$f(x) \equiv a_0 x^n + a_1 x^{n-1} + \dots + a_n = 0$$

make the substitution

$$x = \rho(\cos\theta + i \sin\theta)$$

and separate the resulting equation as before into two equations

$$u \equiv u(\rho,\theta) \equiv a_0\rho^n \cos n\theta + a_1\rho^{n-1}\cos(n-1)\theta + \dots + a_n = 0,$$

$$v \equiv v(\rho,\theta) \equiv a_0\rho^n \sin n\theta + a_1\rho^{n-1}\sin(n-1)\theta + \dots + a_{n-1}\rho\sin\theta = 0.$$

Plot the two equations u = 0 and v = 0,as in the case of rectangular coordinates,by assigning values to ρ and θ and determining intervals in which the curves lie.The roots of f(x) = 0 will be found on the intersection of the two curves.

The chief advantage of this method is that one can determine asymptotes to the curves and use these as starting points for plotting.These curves have been examined carefully[65,20] so that methods of plotting can be greatly simplified by results previously determined.One of these,for example,is the fact that the asymptotes

of either of the curves u = 0 or v = 0 pass through a common point which is the origin when the original equation is transformed so that the coefficient of the (n-1)th power of the variable is zero; and that they divide the plane into n equal sectors.We would expect to find a large value of ρ along a radius vector near one of the asymptotes;while if we found that there were no real point of the curve on two vectors lying in the same sector,we would know that the curve would not occur at any point lying between them.

M.Naraniengar[54] in a resumé of methods of finding complex roots of equations discusses this method and gives plots of the u and v curves for several equations.

CHAPTER IV.

METHODS OF DOUBLE LIMITS.

12. Newton's Method. [1,20,28,56,62,74]

If there is given an approximation γ_1 to a root γ of the equation

$$f(x) = 0,$$

Newton's method consists of finding successive numbers $\gamma_2, \gamma_3, \cdots \gamma_m \cdots$ each after some one γ_k being nearer to the root γ than the preceding one of the sequence, and the limit of the sequence as m becomes infinite being equal to the root. The method is as follows:

If γ is the root and γ_1 is the first approximation to it, designate by γ_1' the difference

$$\gamma_1' = \gamma - \gamma_1$$

By means of Taylor's theorem we have

$$f(\gamma_1+\gamma_1') \equiv f(\gamma_1) + \frac{f'(\gamma_1)}{1!}\gamma_1' + \frac{f''(\gamma_1)}{2!}\gamma_1'^2 + \cdots = 0.$$

If γ_1' is sufficiently small, we may neglect powers of it above the first; and solving the equation above for γ_1' we find :

$$\gamma_1' = -\frac{f(\gamma_1)}{f'(\gamma_1)}$$

Then $\gamma_1 + \gamma_1' = \gamma_2$ will be a new approximation to γ, and can be used in determining a third value γ_3, and the process continued until sufficiently good approximations are obtained.

For complex roots[25] let the approximation be

$$\gamma_1 = \rho_1(\cos\theta_1 + i \sin\theta_1) \equiv \alpha_1 + i\beta_1 .$$

Substituting in the formula given above,

$$\gamma_1' = -\frac{f(\gamma_1)}{f'(\gamma_1)} = -\frac{a_0\gamma_1^n + a_1\gamma_1^{n-1} + \dots + a_{n-1}\gamma_1 + a_n}{na_0\gamma_1^{n-1} + (n-1)a_1\gamma_1^{n-2} + \dots + a_{n-1}},$$

we get the following value of γ_1':

$$\gamma_1' = -\frac{a_0\rho_1^n(\cos n\theta_1 + i\sin n\theta_1) + a_1\rho_1^{n-1}(\cos(n-1)\theta_1 + i\sin(n-1)\theta_1) + \dots + a_n}{na_0\rho_1^{n-1}(\cos(n-1)\theta_1 + i\sin(n-1)\theta_1) + \dots + a_{n-1}}$$

Inserting the numerical values of the sines and cosines,performing the indicated multiplications,collecting terms,and dividing,we find γ_1' in the form:

$$\gamma_1' = \alpha_1' + i\beta_1'$$

Then $\gamma_1 + \gamma_1' = (\alpha_1 + \alpha_1') + i(\beta_1 + \beta_1') \equiv \alpha_2 + i\beta_2 \equiv \gamma_2$.

One disadvantage of the method is that it is long and tedious, as the convergence is often very slow.Another more serious one is that while the method gives a sequence of values approaching the root as a limit,it gives at no time a knowledge of how close to the root the approximation really is.In the case of real roots this difficulty has been overcome ~~for a large number of cases~~ [51] by the union of Newton's method with the Regula Falsi method.By this means two sequences of points are obtained which except for certain special cases approach the root from opposite sides,at least after a certain number of approximations have been made;and thus is obtained an interval between the points of the two sequences in which the root must lie.No such improvement seems to have been made for the case of complex roots.To do so would probably be rather difficult,if possible at all,since the roots may lie anywhere in a plane instead of on a known line.

The conditions for a given approximation leading to any particular root have been carefully worked out for the case of

of real roots and Cayley [14] has done the same for the complex roots of the quadratic equation. Beyond this little seems to have been done in regard to finding regions in which approximations must lie to lead to a particular complex root by Newton's method or in fact to lead to any one of the complex roots of an equation.*

Example: Find the complex root of the equation

$$f(x) \equiv x^3 + x^2 - x + 15 = 0,$$

with a given approximation $\gamma_1 = 3(\cos 60^\circ + i \sin 60^\circ) = 1.5 + 2.6i$.

$$f'(x) = 3x^2 + 2x - 1.$$

$$\gamma_1' = - \frac{27(\cos 180^\circ + i \sin 180^\circ) + 9(\cos 120^\circ + i \sin 120^\circ) - 3(\cos 60^\circ + i\sin 60^\circ) + 15}{27(\cos 120^\circ + i \sin 120^\circ) + 6(\cos 60^\circ + i \sin 60^\circ) - 1}$$

$$= \frac{-27 - 4.5 + 7.8i - 1.5 - 2.6i + 15}{-13.5 + 23.4i + 3 + 5.2i - 1}$$

$$= -0.374 - 0.479i$$

$$\gamma_2 = \gamma_1 + \gamma_1' = 1.5 + 2.6i - 0.374 - 0.479$$

$$= 1.126 + 2.121i = 2.402(\cos 62^\circ 2' + i \sin 62^\circ 2')$$

Use for the new approximation $\gamma_2 = 2.40(\cos 62^\circ 2' + i \sin 62^\circ 2')$.

$$\gamma_2' = - \frac{(2.40)^3(\cos 186^\circ 6' + i \sin 186^\circ 6') + (2.40)^2(\cos 124^\circ 4' + i \sin 124^\circ 4') - (2.40)(\cos 62^\circ 2' + i \sin 62^\circ 2') + 15}{3(2.40)^2(\cos 124^\circ 4' + i \sin 124^\circ 4') + 2(2.40)(\cos 62^\circ 2' + i \sin 62^\circ 2') - 1}$$

$$= -0.113 - 0.109$$

$$\gamma_3 = 1.126 + 2.121 i - 0.113 - 0.109 i \equiv 1.013 + 2.012 i.$$

The true roots are $1 + 2i$, and $1 - 2i$.

13. Horner's Method. [63.]

Given an equation

$$f(x) = 0,$$

with a root $\gamma = \alpha + i\beta$. Assume that the first figures α_1 and β_1 of

*. See, however, Schroder, [64] p.353.

the real and imaginary parts respectively of this root are known. Diminish the roots of the equation by the amount $\alpha+i\beta$. This can be done by synthetic division as in the case of real roots.[63] The coefficients will become complex however. The first figures of the root of the new equation will be equal to the second figures of the root of the original equation. Let them be α_2 and β_2. The roots of the new equation should now be diminished by $\alpha_2+i\beta_2$ and the process continued until a sufficient degree of approximation is reached.

When the root of one of the transformed equations becomes of absolute value less than 1, the first figure of the root can be tentatively obtained very readily from the last two terms of the equation. Assume this equation to be

$$(a'_0+ib'_0)x'^n+\ldots\ldots+(a'_{n-1}+ib'_{n-1})x'+(a'_n+ib'_n)=0.$$

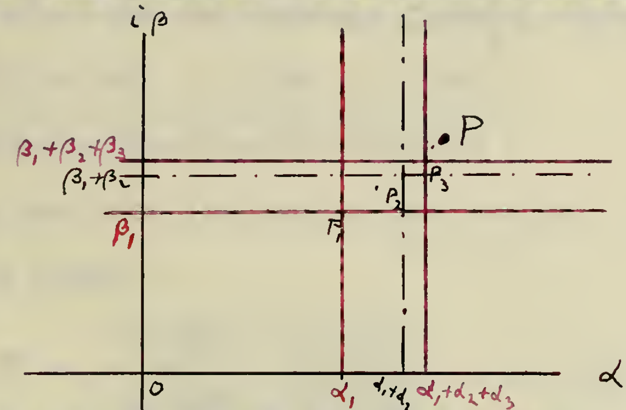
Since the absolute value of the root is less than 1, we neglect powers of x' above the first, and solve for x' from the last two terms, obtaining

$$x' = -\frac{a'_n+ib'_n}{a'_{n-1}+ib'_{n-1}}.$$

If this approximation does not give the exact first figure of the root, the error will be corrected in the process of finding the next succeeding figure of the root.

Geometrically the method is equivalent to the following: Assume a root of the equation at the point P in the plane of the complex variable. Let the point P_1 represent the point whose coordinates are the first figures of the root P; P_2 the point whose coordinates are the first two figures; etc. The successive transformations are equivalent to the transformation of the origin to the points $P_1, P_2, \ldots$ successively,

the axes being translated parallel to themselves. The roots of the successive equations become smaller and smaller in both their real and their imaginary parts; and the root of the original equation is given by the sum of all the real parts, plus the sum of all the imaginary parts, of the roots of the transformed equations.



Example: Find the complex roots of the equation

$$x^4+x^3-5x^2-8x+10=0.$$

From paragraph 6 we find the first figures of the roots to be $-2+i$, and $-2-i$. We proceed to find the former root by diminishing the roots of the equation by $-2+i. \equiv \gamma_1$.

1	+1		-5		-8		+10		$-2+i$
	-2	+i	+1	-3i	+11	+2i	-8	-i	
1	-1	+i	-4	-3i	+3	+2i	+2	-i	
	-2	+i	+4	-7i	+10	+20i			
1	-3	2i	0	-10i	+13	+22i			
	-2	+i	+7	-11i					
1	-5	+3i	+7	-21i					
	-2	+i							
1	-7	+4i							

The first transformed equation, therefore, is as follows:

$$x'^4-(7-4i)x'^3+(7-21i)x'^2+(13+22i)x'+2-i=0.$$

We find x' from the last two terms.

$$\gamma_2 = x' = \frac{-2 + i}{13 + 22i} = 0.00 + 0.08i$$

We diminish the roots of the equation by this amount.

$$\begin{array}{l|l}
1 \ -7+4i \quad +7 \quad +21i \quad +13 \quad +22i \quad +2 \quad -i & \underline{0.00 \quad 0.08i} \\
\quad 0+0.08i+0.3264-0.56i \ +1.7248+0.5339i-1.8027+1.1780i & \\
\hline
1 \ -7 \quad 4.08i \ 6.6736-21.56i+14.7348+22.5339i \ \vert +0.1973+0.1780i & \\
\quad 0+0.08i-0.3328-0.56i \ +1.7696+0.5072i & \\
\hline
1-7 +4.16i+6.3408-22.12i+16.4944+23.0411i &
\end{array}$$

Without carrying the work farther, we can obtain a third approximation from the last two terms.

$$\gamma_3 = x'' = \frac{-0.1973-0.1780i}{16.4944+23.0411i} = -0.009(15) + 0.001(99)i$$

The root of the original equation, therefore, is the sum of the approximations $\gamma_1 + \gamma_2 + \gamma_3 = -2+i + 0.00 + 0.08i - 0.009(15) + 0.001(99)i$

$$= -2.009(15) + 1.081(99)i$$

14. Weddle's Method.

The method of Newton, as described in article 12, assumes the knowledge of an approximation to a root of an equation, and seeks to obtain a better approximation by addition to the known one; that is, if γ_1 is the known approximation, the value $\gamma_1 + \gamma_1'$ (if γ_1' is properly defined) will be a better one. For an infinite number of operations, the method develops into the infinite series

$$\lim_{n \doteq \infty} \gamma_1 + \gamma_1' + \gamma_2' + \cdots + \gamma_n' + \cdots = \gamma.$$

Thomas Weddle has developed a somewhat similar method. Given an approximation γ_1 to a root γ, he seeks a value γ_1'' such that the product $\gamma_1(1+\gamma_1'')$ will be a better approximation to γ than is γ_1. For an infinite number of operations, he gets the infinite product

$$\lim_{n \doteq \infty} \gamma_1(1+\gamma_1'')(1+\gamma_2'') \cdots (1+\gamma_n'') \cdots = \gamma.$$

He proceeds as follows: Let γ be a root of the equation $f(x) = 0$, and assume that γ_1 is an approximation sufficiently near to γ. If we transform the equation by means of the substitution

(1) $$x = \gamma_1 x'$$

we will obtain a new equation $F(x') = 0$ whose root $\gamma' = \frac{\gamma}{\gamma_1}$ will not be much different in absolute value from unity. This root may be written $\gamma' = (1 + \gamma'')$, where " γ'' will be a very small quantity" in absolute value. Expanding by means of Taylor's theorem, we get

$$F(1+\gamma'') \equiv F(1) + F'(1)\gamma'' + \frac{F''(1)}{2!}\gamma''^2 \; . \; . \; . \quad = 0.$$

Neglecting powers of γ'' above the first, we find an approximation to γ'',

$$\gamma_1'' = -\frac{F(1)}{F'(1)}$$

An approximation to the root of the transformed equation, therefore, is $(1+\gamma_1'')$. The new approximation to the root of the original equation, from (1), is $\gamma_1(1+\gamma_1'')$.

The method is satisfactory in certain cases. Difficulties enter, however, in regard to the region in which an approximation must lie in order to lead to a root of the equation. Moreover, if $|\gamma'|$ is approximately 1, then γ'', (defined by the relation $\gamma' = 1 + \gamma''$), is not necessarily very small in absolute value. In fact, for the case where $\gamma' = i$, (and therefore $|\gamma'| = 1$), $|\gamma''|$ is $\sqrt{2}$. Other examples are easily found. These difficulties need to be taken care of before the method can be safely used.

*Example: Given $f(x) = x^3 + x^2 - x + 15 = 0$, and $x = 1.5 + 2.6i$ an approximation to a complex root. Find the root.

$$\gamma_1 = 1.5 + 2.6i \equiv 3(\cos 60^\circ + i \sin 60^\circ) \text{ approximately.}$$

Substituting $x = 3(\cos 60^\circ + i \sin 60^\circ)x'$, we get the equation

$$27(\cos 180^\circ + i \sin 180^\circ)x' + 9(\cos 120^\circ + i \sin 120^\circ)x'$$
$$-3(\cos 60^\circ + i \sin 60^\circ)x' + 15 = 0$$

*This example is given to explain the method of procedure rather than to show the faults of the method. The solution is satisfactory in this special case. In general, the results ~~are~~ seem to be uncertain.

Replacing the cosines and sines by their ~~numerical~~ values and dividing both sides of the equation by -3, we get the new equation

$$F_1(x')=9x'^3+(1.5-2.6i)x'^2+(0.5+0.87i)x'-5=0.$$

$$\gamma_1''\equiv\alpha_1+i\beta_1=-\frac{F_1(1)}{F_1'(1)}=-\frac{9+1.5-2.6i+0.5+0.87i-5}{27+3-5.2i+0.5+0.87i}$$

$$=-0.815+0.033i$$

$$(1+\alpha_1+i\beta_1)=0.815+0.033i\ \cong\ 0.815(\cos 2^\circ 19'+i\sin 2^\circ 19')$$

Substituting $x'=0.815(\cos 2^\circ 19'+i\sin 2^\circ 19')x''$ we get the equation

$$27(0.815)^3(\cos 186^\circ 57'+i\sin 186^\circ 57')x''^3$$
$$+9(0.815)^2(\cos 124^\circ 38'+i\sin 124^\circ 38')x''^2$$
$$-3(0.815)(\cos 62^\circ 19'+i\sin 62^\circ 19')x''+15=0.$$

Replacing the cosines and sines by their ~~numerical~~ values and simplifying, we get the equation

$$(4.835+i\,0.589)x''^3-(1.136-1.638i)x''^2+(0.379+0.724i)x''-5=0.$$

$$\gamma_2''\equiv(\alpha_2+i\beta_2)=-\frac{F_2(1)}{F_2'(1)}=-0.0776+0.0225i.$$

$$(1+\alpha_2+i\beta_2)=0.9224+0.0225i\ \cong\ 0.9224(\cos 1^\circ 24'+i\sin 1^\circ 24').$$

The root of the original equation is given by the product

$$\gamma_1(1+\gamma_1'')(1+\gamma_2'')\equiv\ \gamma_1'(1+\alpha_1+i\beta_1)(1+\alpha_2+i\beta_2)$$
$$=3(\cos 60^\circ+i\sin 60^\circ)(0.815)(\cos 2^\circ 19'+i\sin 2^\circ 19')$$
$$\cdot(0.9224)(\cos 1^\circ 24'+i\sin 1^\circ 24').$$
$$=2.255(\cos 63^\circ 41'+i\sin 63^\circ 41')$$
$$=0.9997+2.021i$$

The true root is $1+2i$.

15. Method of Infinite Series. [49, 50, 41, 42, 43]

E. McClintock [44] has made use of the Laplace series [43] in developing a "method for finding all the roots of an algebraic

equation simultaneously". For the trinomial equation

$$x^n = w^n + nax^{n-k}$$

he derives from the Laplace series the following series:

$$(1) \qquad y = w + w^{1-k}a + w^{1-2k}(1-2k+n)\frac{a^2}{2!} + w^{1-3k}(1-3k+n)\frac{a^3}{3!} + \cdots$$

where y is a root of the trinomial equation, and w is any one of the (n)th roots of w^n.

He gives the following condition as being necessary and sufficient for convergence:

$$|a^n| < |k^{-k}(n-k)^{k-n}\, w^{nk}|, \quad \text{for n always positive.}$$

If for any given equation the series does not converge, he considers the equation in two forms. First, dividing by na and rearranging terms, he gets the equation

$$x^{n-k} = -\frac{w^n}{na} + \frac{x^n}{na}$$

or, placing $\dfrac{-w^n}{na} = w_1$ and $\dfrac{1}{na} = (n-k)a_1$,

$$x^{n-k} = w_1 + (n-k)a_1 x^n.$$

Second, dividing the given equation by x^{n-k}, he gets the equation

$$x^k = na + w^n x^{k-n}.$$

He shows that if the original equation does not give a convergent series, each of these two equations will do so, the first one giving n-k roots, the second the remaining k roots, of the original equation.

For the n-term equation he uses the more general series 50, 76.

$$(2) \quad y = w + w^{1-n}\varphi(w)\cdot a + w^{1-n}\frac{d}{dw}\left[w^{1-n}(\varphi(w))^2\right]\frac{a^2}{2!} + \left[w^{1-n}\frac{d}{dw}\right]^2\left[w^{1-n}(\varphi(w))^3\right]\frac{a^3}{3!} + \cdots$$

where the given equation was

$$x^n = w^n + na\,\varphi(x) \qquad \text{(in general, a = 1).}$$

Having given an equation for solution, his first step is to

For other methods of solving trinomials, see Weber, p.397.

examine it's fitness;that is,the convergency of the series derived from it.A criterion for testing this"fitness"is not given,it being suggested that it can be done by inspection.If the equation is not "fit",a linear transformation exists (so the author states without proof) that will render it "fit".This transformation should be found and applied.

Let us assume that the equation after having been made "fit" is the following:

$$a_0x^n+a_1x^{n-1}+\dots+a_kx^{n-k}+\dots+a_\ell x^{n-\ell}+\dots+a_mx^{n-m}+\dots+a_n=0.$$

The next step is to select "dominant"coefficients.This is done in accordance with the conditions

I. a_0,a_n,are always"dominants".

II."Dominants"must be large in comparison with other coefficients.

III.Three coefficients a_k,a_ℓ,a_m,(coefficients of terms in x^{n-k},$x^{n-\ell}$, x^{n-m} respectively) are called successive"dominants"if $|a_\ell^{m-k}|$ is "decidedly" larger than $|a_k^{\ell-k}\,a_m^{m-\ell}|$. That is,this relation must hold for any set of three successive"dominants".

The "dominants" of the equation having been found to be a_0,a_k, a_ℓ,a_m, ... a_n,the equation is replaced by a set of equations obtained as follows:

Equation 1:Divide the given equation by $a_0x^{n-k}$and rearrange the terms into the form $x^k=\omega^k+ka\varphi_1(x)$.

Equation 2:Divide the given equation by $a_kx^{n-(\ell-k)}$and rearrange the terms into the form $x^{\ell-k}=\omega^{\ell-k}\cdot(\ell-k)a\,\varphi_2(x)$.

Equation 3:Divide the original equation by $a_\ell x^{n-(m-\ell)}$and rearrange the terms into the form $x^{m-\ell}=\omega^{m-\ell}(m-\ell)a\,\varphi_3(x)$.

. . . .

In general,the φ-functions will have negative exponents,and a may be taken equal to 1.

The series[2] above is then applied to each equation. All the roots can be obtained from these equations. The first equation will give a different root for each of the (k)th roots of w^k, giving k roots in all; the second equation will give 1 - k roots; and so forth, the entire set giving therefore all the roots of the original equation.

The method seems to be of considerable practical value. McClintock solves several problems, taking successive ones from a textbook[4] on the Theory of Equations to avoid criticism of having selected special types of examples. If the theoretical foundations could be strengthened by means of criteria for "fitness" of equations, some certain way of selecting suitable transformations, or a more exact method of determining "dominant" coefficients, the work would be of much more value and usefulness.

P. Lambert[41,42] has developed a similar method which also makes use of the Laplace Series[43]. For the three term equation he proceeds as follows:

Let the given equation be

$$f(x) \equiv a_0 x^n + a_{n-k} x^k + a_n = 0 .$$

Form the three equations

$$(1)\ a_0 x^n + a_{n-k} x^k y + a_n = 0,$$

$$(2)\ a_0 x^n + a_{n-k} x^k + a_n y = 0,$$

$$(3)\ a_0 x^n y + a_{n-k} x^k + a_n = 0.$$

These may be written as follows:

$$(1')\ x = \left[-\frac{a_n}{a_0} - y\frac{a_{n-k}}{a_0} x^k \right]^{\frac{1}{n}}$$

$$(2')\ x = \left[-\frac{a_{n-k}}{a_0} - y\frac{a_n}{a_0} x^{-k} \right]^{\frac{1}{n-k}}$$

$$(3')\ x = \left[-\frac{a_n}{a_{n-k}} - y\frac{a_0}{a_{n-k}} x^n \right]^{\frac{1}{k}}$$

Expand each of these functions on the right into a power series in y by means of the Laplace series, which is as follows:

If[41, 43] $$x = f\left[z + y\varphi(x)\right]$$

Then $$F(x) = F(f(z)) + y\varphi(f(z))\frac{d}{dz}\left[F(f(z))\right] - \frac{y^2}{2!}\frac{d}{dz}\left[\overline{\varphi(f(z))}^{2}\frac{d}{dz}\left[F(f(z))\right]\right] + \cdots$$

$$+ \frac{y^n}{n!}\frac{d^{n-1}}{dz^{n-1}}\left[\overline{\varphi(f(z))}^{n}\frac{d}{dz}\left[F(f(z))\right]\right] + \cdots$$

where $F(x)$ represents a root of the equation $f(x)=0$ and $F(f(z))$ is a root of the equation

$$a_0x^n + a_n = 0,$$

the roots of which are assumed to be known.

If after expansion y is made unity, the resulting $F(x)$ becomes one of the roots of the given equation, provided the series is convergent. To each of the three forms above corresponds a series. For each of the n roots of the binomial equation $a_0x^n + a_n = 0$, one of these series will be convergent. Each root will give a different root of the original equation, so that of all n roots are used, all the roots of the orignial equation will be obtained.

The method works out very well for the three term equation, and the conditions for the convergence of the series are simply stated. For equations with more than three terms, however, the process becomes very much involved. The solution, for example, of a six-term equation depends upon the solution of a five-term, a four-term, a three term, and a two-term equation. The solution of the four-term equation demands the convergence of a doubly infinite series, in which every term of an infinite series is an infinite series. A five-term equation demands the convergence of a triply infinite series, and so on.

While theoretically the method may be of some value, the

difficulties presented would seem to be of such a nature as to render it almost useless in practical work for determining the roots of equations of more than three or four terms.

16. Schröder's Method.[64]

E. Schröder[64] made an attempt to generalize the methods of ~~iteration~~ solving equations in order to bring many methods into one group. He proceeds as follows:

Given an equation

$$f(x) = 0$$

and an approximation α_1 to a root α. The problem is to find a function F such that

$$\alpha_2 \equiv F(\alpha_1)$$

will be a better approximation to α than α_1. Then

$$\alpha_3 \equiv F(\alpha_2) \equiv F(F(\alpha_1))$$

will be better than either α_1 or α_2, and in general

$$\alpha_n \equiv F(\alpha_{n-1}) \equiv F(F(\ldots.(F(\alpha_1)\ldots.))$$

will be better than any preceding α_k. The further condition is required that

$$\underset{n \doteq \infty}{L} \alpha_n = \alpha .$$

If we have a function that satisfies these conditions, so that, at least from some point on, each successive application of F gives us a closer approximation to the root α, we have an algorithm for the solution of equations for both real and complex roots.

One example of an F function is given by Newton's method:

$$\alpha_2 = \alpha_1 - \frac{f(\alpha_1)}{f'(\alpha_1)} = F(\alpha_1)$$

$$\alpha_3 = \alpha_2 - \frac{f(\alpha_2)}{f'(\alpha_2)} = F(F(\alpha_1)$$

. . . .

$$\alpha_{n+1} = \alpha_n - \frac{f(\alpha_n)}{f'(\alpha_n)} = F(F(\ldots.F(\alpha_1)\ldots.)).$$

Each successive application of F gives in general a value more nearly equal to the root than any of the preceding approximations, at least from some point on.

The conditions as stated above which the function F must satisfy can be more compactly stated as follows:

I. $|\alpha-\alpha_1| > |\alpha-\alpha_2| > |\alpha-\alpha_3| > \cdots > |\alpha-\alpha_n| > \cdots$

II. $\underset{n \doteq \infty}{L} |\alpha - \alpha_n| = 0.$

It is obvious, also, that we must have from I

I'. $F(\alpha) = \alpha$.

If we expand

$$F(\alpha+\varepsilon) = F(\alpha) + \varepsilon \cdot F'(\alpha) + \frac{\varepsilon^2}{2!}F''(\alpha) + \ldots.$$

and if we place $F(\alpha) = \alpha$ and $F(\alpha+\varepsilon) = \alpha_1$, we may disregard powers of ε above the first, obtaining the relation

$$\alpha_1 = \alpha + \varepsilon \cdot F'(\alpha)$$

or, $\varepsilon \cdot F'(\alpha) = \alpha_1 - \alpha$,

which is equivalent to saying that if the new correction $\alpha_1-\alpha$ is to be in absolute value less than the original correction ε, the following relation must hold:

II'. $|F'(\alpha)| < 1.$

Thus, in the case of Newton's method, where

$$F(\alpha_2) = \alpha_1 - \frac{f(\alpha_1)}{f'(\alpha_1)}$$

we have (since $f(\alpha) = 0$)

I'. $F(\alpha) = \alpha$,

$$\text{II}'.\ F'(\alpha) = \frac{d\left(\alpha - \frac{f}{f'}\right)}{d\alpha} = 1 - \left(\frac{f'^2 - ff''}{f'^2}\right) = 1 - 1 = 0.$$

Schröder also takes up other algorithms for the solution of equations which include a number of other methods as special cases. One of these is the following:

$$F(\alpha_2) = \alpha_1 - \frac{\alpha_1 f(\alpha_1)}{\alpha_1 f'(\alpha_1) - \lambda f(\alpha_1)}$$

where λ is a parameter which can take any value. If λ becomes zero, we see that the algorithm reduces to Newton's method. Another algorithm is

$$F(\alpha_2) = \alpha_1 - \frac{\alpha_1 f(\alpha_1) . f'(\alpha_1)}{\alpha_1 \left[f'^2(\alpha_1) - f(\alpha_1) f'(\alpha_1)\right] - \lambda f(\alpha_1) f'(\alpha_1)}.$$

Schröder obtains another class of methods by combining an infinite number of operations into one step. These methods are not discussed in this thesis on account of the difficulty of clearly presenting the matter in brief. It also seems that their practical value may be less than their theoretical value, although several important known methods (such as Newton's method) are obtained from the general method by specialization.

17. Additional Methods.

A number of other methods for solving equations with complex roots have been developed. One in particular that should be mentioned is the Fürstenau method of infinite determinants. Günther,[32, 33], about 1870, examined this method and developed it to some extent. Concerning his first article, the "Jahrbuch über die Fortschritte der Mathematik" says:[36] "Der daraus gezogene Schluss, dass der wahre Werth der Wurzel zwischen zwei aufeinanderfolgenden Näherungswerthen liege, ist aber nicht begründet, da u.A. der entsprechende Satz nicht von allgemeinen

Kettenbrüchen gilt." Later, in 1877, Günther[32] solves the equation

$$(x+1)^4 \equiv x^4+4x^3+6x^2+4x+1 = 0$$

and finds a root to be $-\frac{2}{3}$ approximately, which he says is a good approximation since it reduces the left-hand side to 0.012 . Naegelbach[52] has also discussed the method and has solved several problems, but the process seems too complicated for practical purposes.

Gonggrijp[30] has recently published a method related to Gräffe's method which seems to offer difficulties in the determination of successive derivatives. These appear to make it impractical for general use, though it may have considerable theoretical value.

Naraniengar[54] gives a geometrical interpretation to several methods that have been discussed in this thesis.

Weber[72] gives a method in connection with his proof of the Fundamental Theorem of Algebra which is of theoretical importance rather than of practical value. It consists in finding circles in which roots lie, and finding smaller circles within these which contain roots, and so forth. It is given to show that the Fundamental Theorem proof leads to a method of finding roots as well as proving their existence.

Runge,[61] Cohn,[16] and others have given methods based on the Theory of Analytic Functions of a Complex Variable which could not be treated here. The problem of separation of roots (that is, the determination of intervals or areas in which only one root lies) is too extensive to be discussed in this article. Graphical methods,[46, 63, 54] , several electro-chemical methods[47, 28, 48, 53] based upon work by Lucas, and a mechanical method developed by L. Torres[28] lie beyond the limits of this thesis.

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*Murphy's Algebraical Equations contains an explanation of the work of Simpson.

In general,only the first page of the discussion of interest to this thesis is given.